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Targeting chaos through adaptive control

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We describe adaptive control algorithms whereby a chaotic dynamical system can be steered to a target state with desired characteristics. A specific implementation considered has the objective of directing the system to a state which is more chaotic or mixed than the uncontrolled one. This methodology is easy to implement in discrete or continuous dynamical systems. It is robust and efficient, and has the additional advantage that knowledge of the detailed behavior of the system is not required. [S1063-651X(98)50503-X]

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Adaptive control algorithms have hitherto been implemented for the purpose of maintaining periodic behavior in nonlinear systems [1–3]. Recently there has been some interest in control algorithms whose aim is to target other types of dynamical behavior. ''Anticontrol'' algorithms, namely, those wherein the objective is to maintain [4] or to *enhance* [5] the chaoticity of dynamical systems have been devised. These efforts have been motivated by practical situations where the enhancement or maintenance of chaos has desirable consequences. Examples of these can be found in contexts as diverse as mixing flows [6], electronic systems [7], and chemical reactions [8], where the enhancement of chaos can lead to improved performance, or in biological applications such as in neural systems, where the maintenance of chaos provides the key to the avoidance of pathological behavior [9].

In this Rapid Communication we describe an adaptive anticontrol algorithm which is simple and easily implemented. The algorithm is set up to maintain a desired level of chaoticity, and to achieve a target value of the local Lyapunov exponent or a local stretching rate. The technique is sufficiently general and can be extended so as to make a given dynamical system achieve a target value of any desired variable or function.

In the context of nonlinear dynamical systems, the method of adaptive control [1,2] applies a feedback loop in order to drive the system parameter (or parameters) to the values required so as to achieve a desired or target state. This is implemented by augmenting the evolution equation for the dynamical system by an additional equation for the evolution of the parameter(s) as described below.

Consider a general N-dimensional dynamical system described by the evolution equation

$$\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X};\mu;t), \tag{1}$$

where $\mathbf{X} \equiv (X_1, X_2, \ldots, X_N)$ are the state variables and μ is the parameter whose value determines the nature of the dynamics. The adaptive control is effected by the additional dynamics

$$\dot{\mu} = \epsilon(\mathcal{P}^\star - \mathcal{P}), \tag{2}$$

where $\mathcal{P}^\star$ is the target value of some variable or property (which could be a function of several variables) $\mathcal{P}$, and the value of ϵ indicates the stiffness of control. The extension to the situation of several control parameters is straightforward.

The scheme is adaptive since in the above procedure the parameters which determine the nature of the dynamics *self-adjust* or adapt themselves to yield the desired dynamics. This has also been termed ''dynamic feedback control'' in the literature [10]. The adaptive principle is remarkably robust and efficient in generic nonlinear systems [2], and may therefore be of considerable utility in a large variety of phenomena, ranging from biological units to control engineering.

For the maintenance of a stable fixed point [1] in a discrete dynamical system for example, the procedure is as follows. The nonlinear system evolves according to the appropriate equation

$$x_{n+1} = f(\alpha, x_n), \tag{3}$$

where α is the parameter to be controlled. If the required value of x is, say, $x^\star$, then the additional equation (for $\mathcal{P} \equiv x$)

$$\alpha_{n+1} = \alpha_n + \epsilon(x^\star - x_n) \tag{4}$$

has the desired effect of tuning the value of α so that the dynamics of the combined equations gives $x \to x^\star$ over a wide range of initial conditions. The stiffness ϵ determines how rapidly the system is controlled. The control time, defined as the time required to reach the desired state, is crucially dependent on the value of ϵ. Numerical experiments show that for small ϵ the recovery time is inversely proportional to the stiffness of control. This follows from the fact that when ϵ is small compared to the inverse timescales in the original dynamical system, we can use an adiabatic approximation since $\dot{\mu} \to 0$, from which [10] it follows that control time will be proportional to $1/\epsilon$. With modifications,

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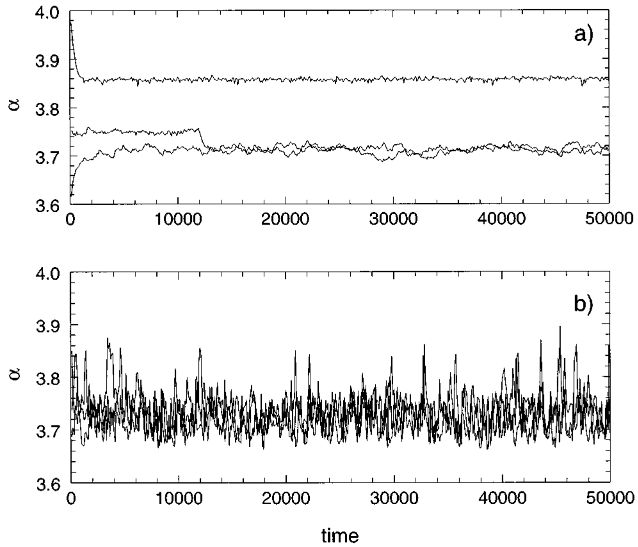

FIG. 1. Variation of the parameter α as a function of iteration step. The target Lyapunov exponent is $\lambda^\star = 0.36$, and the stiffness is (a) $\epsilon = 0.001$ and (b) $\epsilon = 0.01$. The different curves correspond to different initial α. In all cases, the target $\lambda^\star$ is achieved rapidly and maintained.

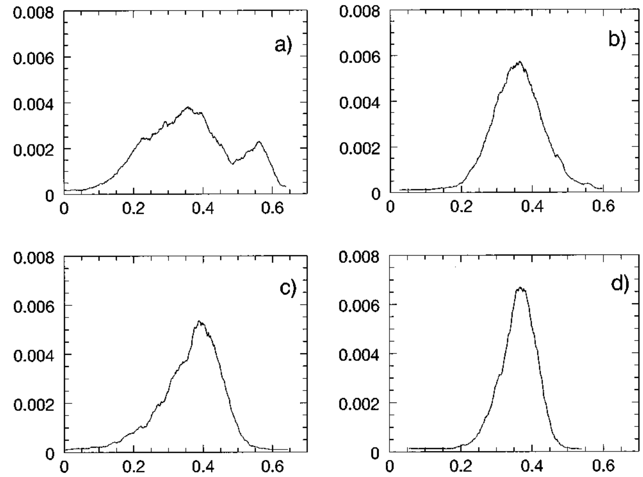

FIG. 2. Probability distribution for finite time or N-step Lyapunov exponents in the adaptively controlled system. The target Lyapunov exponent is $\lambda^\star = 0.36$ as in Fig. 1. (a) $N = 20$ and (b) $N = 50$ correspond to a stiffness of $\epsilon = 0.001$, while (c) $N = 20$ and (d) $N = 50$ correspond to a stiffness of $\epsilon = 0.01$.

this method can be made to control to a stable or unstable periodic orbit of arbitrary period [2,10,11].

If the desired target state is chaotic rather than periodic, one needs to choose an appropriate property $\mathcal{P}$ which should reflect the desired chaotic nature of the target state. Therefore the natural choice of $\mathcal{P}$ is the Lyapunov exponent. It is thus clear that in order to achieve a desired value of the Lyapunov exponent, say, $\lambda^\star$, the procedure to be followed is similar (with $\mathcal{P} \equiv \lambda$).

For a one-dimensional (1D) discrete dynamical system as in Eq. (3) above, the Lyapunov exponent is defined through

$$\lambda = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \ln|f'(\alpha, x_i)|. \tag{5}$$

The control equation (4) takes the form

$$\alpha_{n+1} = \alpha_n + \epsilon(\lambda^\star - \lambda_n), \tag{6}$$

where $\lambda_n = \ln|f'(\alpha, x_n)|$ is the instantaneous value of the Lyapunov exponent. Implementation of the methodology in, say, the logistic equation, is direct and the relevant equations are

$$x_{n+1} = \alpha_n x_n (1 - x_n), \tag{7}$$

$$\lambda_n = \ln|\alpha_n(1 - 2x_n)|, \tag{8}$$

$$\alpha_{n+1} = \alpha_n + \epsilon(\lambda^\star - \lambda_n). \tag{9}$$

Shown in Fig. 1 is an implementation of the control for $\lambda^\star = 0.36$. Since $\lambda(\alpha)$ for the logistic equation is a highly nonmonotonic function, there can be several parameter values for which the system has the same λ, namely, several different attractors with the same Lyapunov exponent are possible. For example, the Lyapunov exponent is approximately 0.36 at $\alpha \approx 3.7$ as well as at $\alpha \approx 3.86$. Which of these values the system adaptively goes to depends on the initial state, the stiffness, and the effective basin of attraction (in parameter space). For small stiffness, the system sticks closely to one or the other attractor, only occasionally making an excursion from one to the other, while for large stiffness, the fluctuations in the parameter are much larger, as can be seen in Fig. 1. For small values of stiffness the time taken to reach the desired goal is usually inversely proportional to the stiffness of control. Note, however, that increasing stiffness beyond a point can make the method unstable and the dynamics unbounded. There is, therefore, an optimal strategy to be employed. While the optimal strategy to be used can be worked out easily in a practical implementation, an analytic optimality criterion is difficult to define.

The distribution of finite-time Lyapunov exponents shows that the short-time chaoticity properties of the adaptive system can be quite different from the equivalent chaotic system. Shown in Fig. 2 are the distributions of adaptively controlled systems with the above average $\lambda = 0.36$, and with different stiffness, for 20 and 50 steps, respectively. While the desired λ is maintained in all cases, it is clear that the adaptation works differently for large or small stiffness. Low stiffness allows the system to explore different attractors with different properties, giving a wider spread in the Lyapunov exponents, while a higher stiffness ensures that the local $\lambda \sim \lambda^\star$, narrowing the distribution. Extensions of this procedure can be made to control higher-dimensional systems.

The fact that the control is always operative means that the augmented system is robust to perturbations. Indeed, if the parameter is perturbed to a very different value, the system readily and rapidly recovers to a dynamics such that the Lyapunov exponent is (nearly) $\lambda^\star$, again with time that inversely depends on ϵ. Note, however, that this control works only in the case of positive $\lambda^\star$: one cannot adaptively control in this manner to a periodic orbit.

An application of practical importance is in enhancing the mixing in chaotic systems. The appropriate adaptive strategy then is to take $\mathcal{P}$ to be the stretching rate. Equation (2) thus becomes

$$\dot{\mu}=\epsilon(\mathcal{R}_{\text{target}}-\mathcal{R}_{\text{local}}), \tag{10}$$

where $\mathcal{R}_{\text{local}}$ is the instantaneous local stretching rate, and $\mathcal{R}_{\text{target}}$ is the prescribed desired stretching rate, which can in principle be in any one of the dynamical variables characterizing the system.

As an example, consider the Lorenz attractor,

$$\begin{aligned}\dot{x}&=\sigma(y-x),\\ \dot{y}&=\mu x-y-xz,\\ \dot{z}&=xy-bz.\end{aligned} \tag{11}$$

Choosing the evolution equation for the parameter to be

$$\dot{\mu}=\epsilon(\mathcal{R}_{\text{target}}-\dot{x}), \tag{12}$$

where the instantaneous stretching rate $\mathcal{R}_{\text{local}}=\dot{x}$, with $\dot{x}$ given by the evolution equation (11), achieves the objective. Note that instead of the x direction, y or z can equally effectively be used in the above control.

In the absence of knowledge of the evolution equation the above control can be effected by the discrete evolution equation

$$\mu_{t+\delta t}=\mu_t+\epsilon(\mathcal{R}'_{\text{target}}-\Delta x_t), \tag{13}$$

where Δx_t is the local stretching given by $\|x_t-x_{t-\delta t}\|$ (δt small) where $\mathcal{R}'_{\text{target}}=\delta t\mathcal{R}_{\text{target}}$.

Shown in Fig. 3 is the result of an implementation of this adaptive anticontrol equation (13) where the target stretching rate is specified to be 1.0 with the control stiffness $\epsilon=0.1$. As can be seen, the controlled parameter first rapidly climbs (the rate of ascent being directly proportional to $1/\epsilon$) from an initial value $\mu=35.0$. Around $\mu\sim380$, it settles into fluctuations which are of the integrated white noise type [i.e., the power spectrum of the time series of these fluctuations is clearly $S(f)\sim1/f^2$] lead to a very mixed system. Starting off with any other value of μ leads to the same result, as does control via Eq. (12) using the relation between $\mathcal{R}_{\text{target}}$ and $\mathcal{R}'_{\text{target}}$ defined above.

The nature of adaptive anticontrol is such that the dynamics that obtains is intrinsically mixing. In contrast with similar techniques where a stable state is targetted, the present mechanism [12] essentially drags the system rapidly to the first appropriate state encountered in parameter space, namely, one which matches the targeted local stretching rate. The system, in effect moves from (chaotic) attractor to attractor, with significant fluctuations in the parameter that is being controlled. By choosing a small target stretching rate, this same control mechanism can be used to obtain cycles as well. Somewhat fuzzy cycles are obtained, with the fuzziness decreasing with decreasing ϵ.

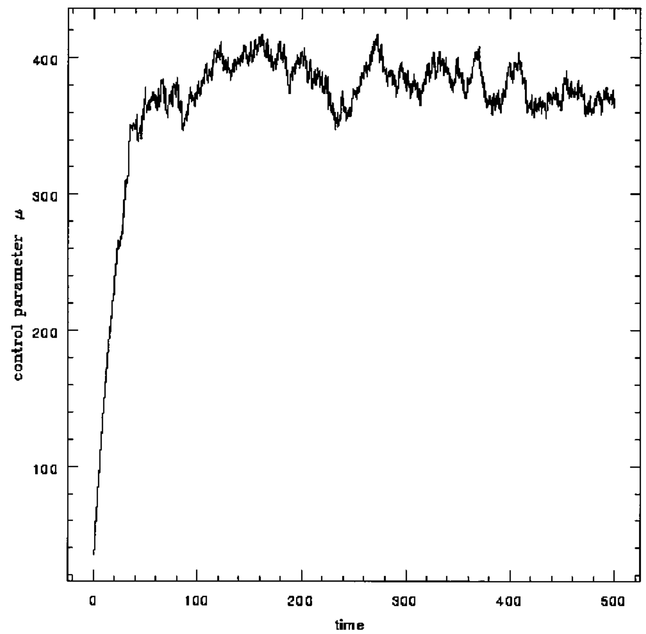

FIG. 3. Variation of the parameter μ as a function of time for the Lorenz attractor. The target stretching rate is 1.0, the stiffness is $\epsilon=0.1$, and the time step is $dt=0.01$.

It should be emphasized that knowledge of the map in the control algorithm above is in principle not required, since the necessary information required to implement adaptive anticontrol is simply the difference between the current value of the variable and its previous value. On the other hand, it is essential that the parameter μ being controlled must have the driving power in order to effect large dynamical changes. Parameters which are suitable for controlling are easy to identify through the appropriate dynamical phase diagrams.

In summary, we have presented here an adaptive algorithm which can be used to achieve desired chaotic behavior in nonlinear dynamical systems. The anticontrol technique, which is rapid, powerful and robust, extends adaptive control methods for obtaining periodic orbits [1,2,11]. We have applied this to the case of achieving a target value of the Lyapunov exponent, or a desired value of the local stretching rate and found that the methodology is successful in a number of examples, including multidimensional and multiparameter systems.

An important consideration is that the present method can be implemented without explicit knowledge of the dynamics. The possibility of treating the system as a black box is likely to be of utility in complex experimental applications [8,9] necessitating the controlled maintenance or enhancement of chaos.

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[12] While the sign of the control dynamics given above holds in generic cases, there may be situations where the sign of the control dynamics is negative (for instance in the case of parameters σ and b in the Lorenz system). This will happen if the value of the initial parameter is larger than the closest parameter which matches the target (not smaller, as is usual). In real implementations (where the dynamics is a black box) a gross scan of the parameter space will give a reasonable indication as to which sign to use.